L28 March 24 Htpy

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Continuous Change

Let X, Y be spaces. Two continuous maps
$f, g : X \longrightarrow Y$ are homotopic if
$\exists$ continuous $H : X \times [0,1] \longrightarrow Y$, call homotopy
such that $\left.\begin{array}{l} H(x,0) = f(x) \\ H(x,1) = g(x) \end{array}\right\} \ \forall\ x \in X$

Notation. $f \simeq g$ or $f \overset{H}{\simeq} g$

Example

① A rotation R_α on $\mathbb{R}^2$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto R_\alpha(x) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Any two R_α, R_β are homotopic

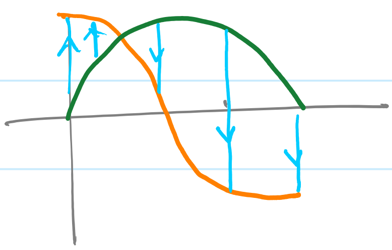
e.g $H(x,t) = R_{(1-t)\alpha + t\beta}(x)$

Clearly, homotopy may not be unique.

② $f, g : [0, \pi] \longrightarrow \mathbb{R}^2$

$f(x) = \sin x$

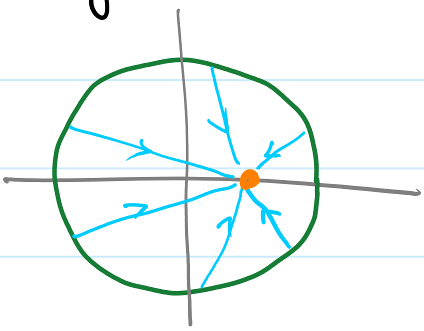
$g(x) = \cos x$



$H(x,t) = \sin\left(x + \frac{t\pi}{2}\right)$

③ $X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, $Y = \mathbb{R}^2$

$f, g : S^1 \longrightarrow \mathbb{R}^2$, $f(x+iy) = (x, y)$, $g(z) = (\frac{1}{2}, 0)$



$H(z, t) = (1-t) z + \frac{1}{2}$

④ $X = S^1$, $Y = \mathbb{R}^2 \setminus \{(0,0)\}$

For the above f, g, $f \not\simeq g$.

Null homotopic

A map $c : X \longrightarrow Y$ with $c(x) = y_0 \ \forall x \in X$ is called a constant map (onto $y_0 \in Y$)

If $f : X \longrightarrow Y$ satisfies $f \simeq c$ then f is null homotopic or homotopically trivial.

Fact. Any map $f : X \longrightarrow \mathbb{R}^n$, $n \geq 1$, is null homotopic.

$H : X \times [0,1] \longrightarrow \mathbb{R}^n$, $H(x,t) = (1-t) f(x)$

Example $Y \subset \mathbb{R}^n$ is called star-shaped if

$\exists y_0 \in Y \ \forall y \in Y \ \underbrace{\{(1-t)y + t y_0 : t \in [0,1]\}}_{\text{straight line joining } y \text{ to } y_0} \subset Y$

Qu. Can we replace the straight lines by other continuous paths?

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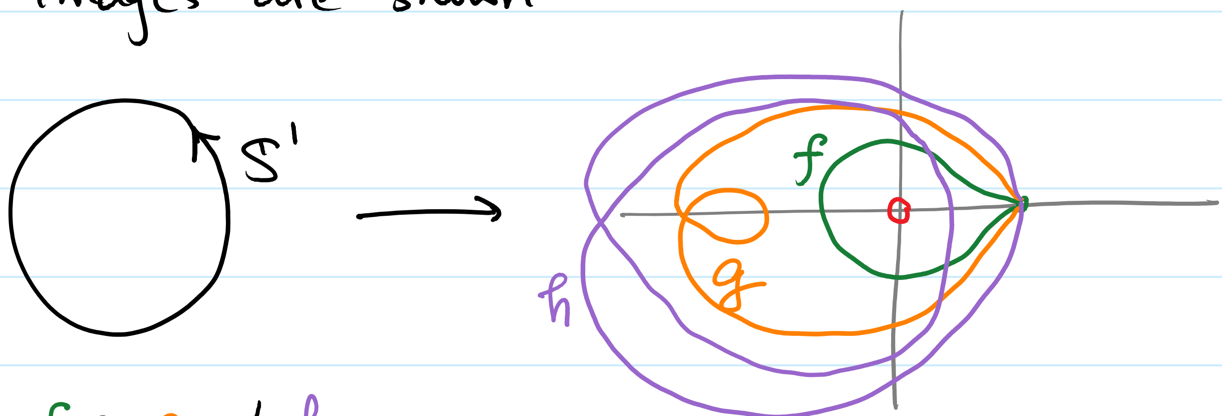
Example. Consider the following three maps $f, g, h : S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$. Their images are shown



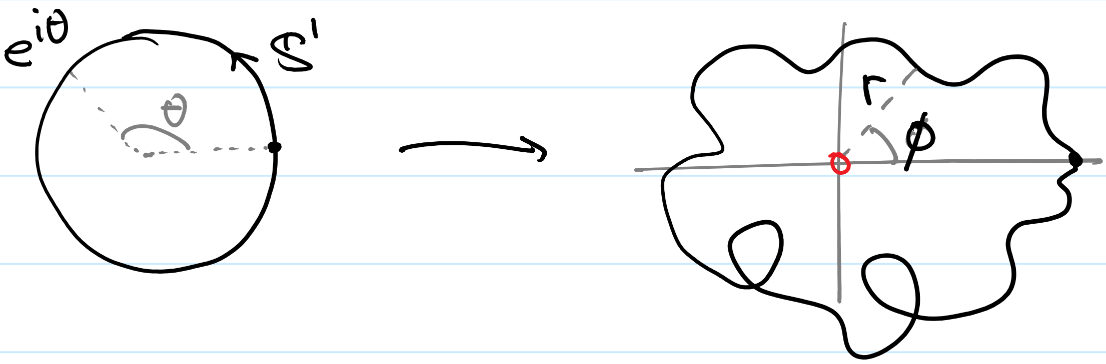
$f \simeq g \not\simeq h$

This can only be understood intuitively now.

Intuition

For any map $S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$, it can be expressed as $e^{i\theta} \longmapsto re^{i\phi}$ where

$$r = r(\theta) > 0 \quad \text{and} \quad \phi = \phi(\theta)$$



We only need to worry about ϕ because any two $r_1, r_2 > 0$ can be easily homotopic.

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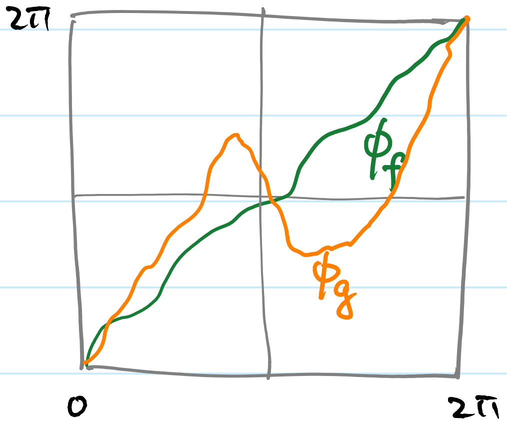
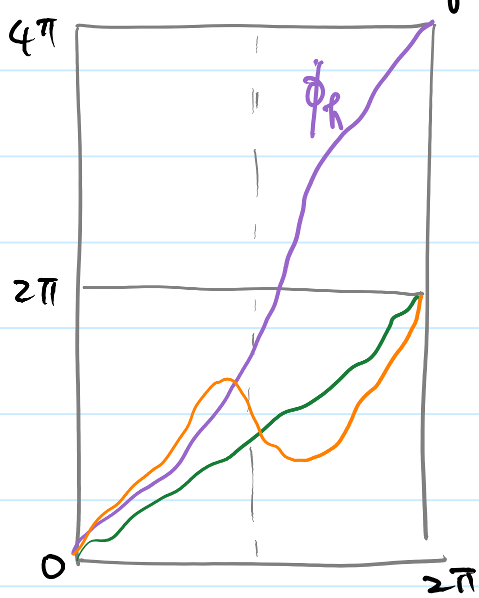
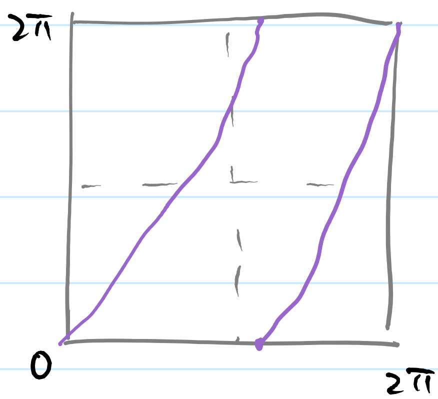
Without loss of generality, assume $\phi(0)=0$.
Then, when one varies θ in the domain S^1, $\phi=\phi(\theta)$ changes dependantly continuously.
For the example of f and g, the graphs of ϕ are drawn below



Note that the two "ends" at $(0,0)$ and $(2\pi,2\pi)$ actually correspond to the same point on the loops of f and g.

In the above pictures, it is easy to continuously change ϕ_f to ϕ_g with the two end-points fixed. This gives a homotopy between f and g.
However, the graph of h is different.





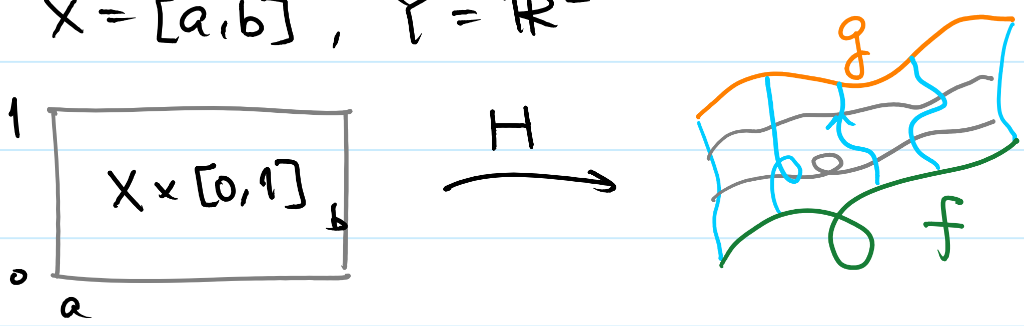
If we expect ϕ_h goes from $(0,0)$ to $(2\pi, 2\pi)$, we can only have the discontinuous graph shown on the right hand side picture.

To have a continuous ϕ_h, the graph goes from $(0,0)$ to $(2\pi, 4\pi)$.

One cannot at the same time fixed the end-points and continuously change ϕ_h to any of ϕ_f or ϕ_g.

Pictures

We usually draw pictures of homotopy by
$X = [a,b]$, $Y = \mathbb{R}^2$



Given spaces X, Y, the homotopy relation $\simeq$ is an equivalence relation on maps: $X \longrightarrow Y$

Reflexive

$f \simeq f$ by $H(x,t) = f(x)$ $\forall t \in [0,1]$

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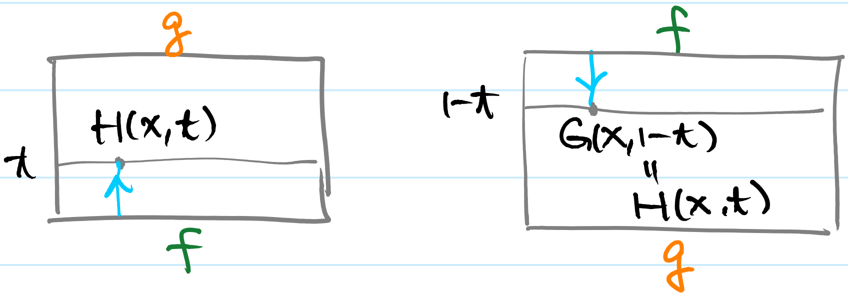
Symmetric Let $f \overset{H}{\simeq} g$, Then $g \overset{G}{\simeq} f$

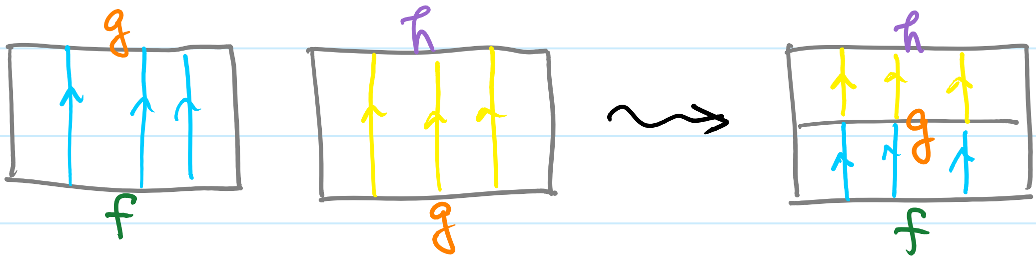
where $G(x,t) = H(x, 1-t)$



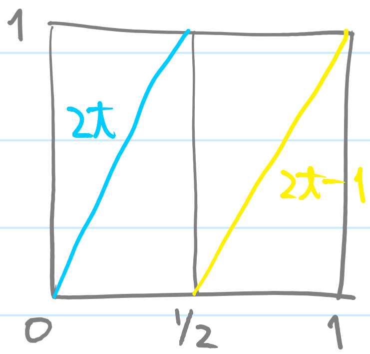
Transitive Let $f \overset{F}{\simeq} g$, $g \overset{G}{\simeq} h$ Then

$\exists\ H : X \times [0,1] \longrightarrow Y$, $f \overset{H}{\simeq} h$

$$H(x,t) = \begin{cases} F(x, 2t) & t \in [0, \frac{1}{2}] \\ G(x, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$



Adjust the clock



Conclusion

Let $C(X,Y)$ = the set of continuous maps $X \to Y$

$[X,Y] = C(X,Y)/\simeq$

↑ set of homotopy classes

Example. $[S^1, Y]$ reflects some topological structure of the space Y.

$[S^1, \mathbb{R}^n] = \{[c]\}$, singleton class of the constant map
$n \geq 1$

$[S^1, \mathbb{R}^2 \setminus \{0\}] \neq$ singleton $[f] = [g] \neq [h]$

$\therefore [S^1, \mathbb{R}^2 \setminus \{0\}] \neq [S^1, \mathbb{R}^n]$, $n \geq 1$

Expect $\mathbb{R}^2 \setminus \{0\} \neq \mathbb{R}^n$, $n \geq 1$

Theorem you need!

* If $Y_1 = Y_2$ (homeomorphic) then $\forall$ space X, $[X, Y_1], [X, Y_2]$ are **bijective**
* If $X_1 = X_2$ (homeomorphic) then $\forall$ space Y, $[X_1, Y], [X_2, Y]$ are **bijective**

Idea of proof

Let $\varphi : Y_1 \longrightarrow Y_2$ be a homeomorphism

Define $\varphi_\# : [X, Y_1] \longrightarrow [X, Y_2]$ by

$$\underset{: X \to Y_1}{[f]} \longmapsto \underset{: X \to Y_1 \to Y_2}{[\varphi \circ f]}$$

Qu. What do we need to check?

(i) $\varphi_\#$ is well-defined, i.e. $[f] = [g] \Rightarrow [\varphi f] = [\varphi g]$

(ii) 1-1

(iii) onto

$\left.\begin{matrix}\text{(ii) 1-1}\\ \text{(iii) onto}\end{matrix}\right\}$ Hope: $(\varphi_\#)^{-1} = (\varphi^{-1})_\#$

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More about the proof

(i) $[f]=[g] \Rightarrow [\varphi\circ f]=[\varphi\circ g]$

$$\begin{array}{ccc} [f]=[g] & & [\varphi\circ f]=[\varphi\circ g] \\ \Updownarrow & & \Updownarrow \\ f\simeq g & & \varphi\circ f\simeq \varphi\circ g \end{array}$$

(ii), (iii) $(\varphi_\#)^{-1}=(\varphi^{-1})_\#$ $\quad [f] \xrightarrow{\varphi_\#} [\varphi\circ f] \xrightarrow{(\varphi^{-1})_\#} [\varphi^{-1}\circ\varphi\circ f]$

More general,

$$\begin{array}{ccccc} [f] & \xrightarrow{\varphi_\#} & [\varphi\circ f] & \xrightarrow{\psi_\#} & [\psi\circ\varphi\circ f] \\ \| & & & & \| \\ [g] & \longrightarrow & [\varphi\circ g] & \longrightarrow & [\psi\circ\varphi\circ g] \end{array}$$

The crucial argument used in (i), (ii), (iii) is the result below.

Ultimate Theorem Let X, Y, Z be spaces and

$$X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \underset{g_1}{\overset{g_0}{\rightrightarrows}} Z.$$ If $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$

then $g_0\circ f_0 \overset{H}{\simeq} g_1 f_1 \simeq g_1 f_0 \simeq g_0 f_1 : X \longrightarrow Z$

Construct $H: X\times[0,1] \longrightarrow Z$ by

$$H(x,t)=G\big(F(x,t),t\big)$$

The other homotopies are similar

Example. Let us consider

$$X = S^0 = \{ x \in \mathbb{R} : |x|^2 = 1 \} = \{\pm 1\} \subset \mathbb{R}$$

Qu. What is the meaning of $[S^0, Y]$?

(i) For $[f], [g] \in [S^0, Y]$, we have

$$f, g : S^0 = \{\pm 1\} \longrightarrow Y$$

There are four points

$$f(-1), f(1), g(-1), g(1) \in Y$$

$f \simeq g \Leftrightarrow \exists$ two paths joining

$f(-1)$ to $g(-1)$; $f(1)$ to $g(1)$

If Y is path connected then $f \simeq g$ $\forall f, g$

$\therefore [S^0, Y]$ is singleton

Qu. What if Y has two path components?

For convenience, we consider mappings of a pair $(S^0, -1) \longrightarrow (Y, y_0)$

i.e., $f : S^0 \longrightarrow Y$ such that $f(-1) = y_0$

Then $[(S^0, -1), (Y, y_0)]$ exactly counts the number of path components of Y.

Mappings of a pair

Let $A \subset X$, $B \subset Y$, $f : (X, A) \longrightarrow (Y, B)$ is a continuous mapping $f : X \longrightarrow Y$ such that $f(A) \subset B$.